

# **TOPOLOGICAL PROPERTY OF THE HOLONOMY DISPLACEMENT ON THE PRINCIPAL $U(n)$ -BUNDLE OVER $D_{n,m}$ , RELATED TO COMPLEX SURFACES**

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**ABSTRACT.** Consider  $D_{n,m} = U(n, m) / (U(n) \times U(m))$ , the dual of the the Grassmannian manifold and the principal  $U(n)$  bundle over  $D_{n,m}$ ,  $U(n) \rightarrow U(n, m) / U(m) \xrightarrow{\pi} D_{n,m}$ . Given a nontrivial  $X \in M_{m \times n}(\mathbb{C})$ , consider a two dimensional subspace  $\mathfrak{m}' \subset \mathfrak{m} \subset \mathfrak{u}(n, m)$ , induced by  $X, iX \in M_{m \times n}(\mathbb{C})$ , and a complete oriented surface  $S$ , related to  $(X, g) \in M_{m \times n}(\mathbb{C}) \times U(n, m)$ , in the base space  $D_{n,m}$  with a complex structure from  $\mathfrak{m}'$ . Let  $c$  be a smooth, simple, closed, orientation-preserving curve on  $S$  parametrized by  $0 \leq t \leq 1$ , and  $\hat{c}$  its horizontal lift on the bundle  $U(n) \rightarrow U(n, m) / U(m) \xrightarrow{\pi} D_{n,m}$ . Then the holonomy displacement is given by the right action of  $e^\Psi$  for some  $\Psi \in \text{Span}_{\mathbb{R}}\{i(X^* X)^k\}_{k=1}^q \subset \mathfrak{u}(n)$ ,  $q = \text{rk} X$ , such that

$$\hat{c}(1) = \hat{c}(0) \cdot e^\Psi \quad \text{and} \quad \text{Tr}(\Psi) = 2i \text{Area}(c),$$

where  $\text{Area}(c)$  is the area of the region on the surface  $S$  surrounded by  $c$ , obtained from a special 2-form  $\omega_{(X,g)}$  on  $S$ , called an area form  $\omega_{(X,g)}$  related to  $(X, g)$  on  $S$ .

## 1. INTRODUCTION

Gauss-Bonnet Theorem shows a kind of relation between Riemannian Geometry and Topology through two kinds of curvatures -Gaussian curvature and geodesic curvature-(, angles if needed) and Euler-characteristic. In this paper, we explain a similar phenomenon in some principal bundles through area and holonomy displacement.

In [6], Pinkall showed that the holonomy displacement of a simple closed curve in the base space on the Hopf bundle  $S^1 \rightarrow S^3 \rightarrow S^2$  depends on the area of its interior. Byun and Choi [1] generalized this result to the principal  $U(n)$ -bundle over the Grassmannian manifold  $G_{n,m}$  of complex  $n$ -planes in  $\mathbb{C}^{n+m}$ ,

$$U(n) \rightarrow U(n + m) / U(m) \rightarrow G_{n,m},$$

by introducing  $U_{m,n}(\mathbb{C})$ , where

$$U_{m,n}(\mathbb{C}) := \{X \in M_{m \times n}(\mathbb{C}) \mid X^* X = \lambda I_n \text{ for some } \lambda \in \mathbb{C} - \{0\}\}.$$

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Especially, the result related to a complex surface in  $G_{n,m}$  can be summarized as follows:

**Theorem 1.1.** [1] *Assume  $U(k)$ ,  $k = 1, 2, \dots$ , has a metric, related to the Killing-Cartan form, given by*

$$\langle A, B \rangle = \frac{1}{k} \operatorname{Re}(\operatorname{Tr}(A^* B)), \quad A, B \in \mathfrak{u}(k),$$

*Consider a bundle  $U(n) \rightarrow U(n+m)/U(m) \xrightarrow{pr} G_{n,m}$ , where  $pr : U(n+m)/U(m) \rightarrow G_{n,m}$  is a Riemannian submersion. Given a nontrivial  $X \in U_{m,n}(\mathbb{C})$ , a two dimensional subspace  $\mathfrak{m}' \in \mathfrak{u}(n+m)$ , induced by  $X$  and  $iX$ , gives rise to a complete totally geodesic surface  $S$  with a complex structure in the base space  $G_{n,m}$ . And if  $c : [0, 1] \rightarrow S$  is a piecewise smooth, simple, closed curve on  $S$  and if  $\tilde{c}$  is its horizontal lift, then the holonomy displacement along  $c$ ,*

$$\tilde{c}(1) = \tilde{c}(0) \cdot V(c),$$

*is given by the right action of  $V(c) = e^{i\theta} I_n \in U(n)$ , where  $A(c)$  is the induced area of the region, surrounded by  $c$ , on the surface  $S$ , from the metric on  $G_{n,m}$ , and  $\theta = 2 \cdot \frac{n+m}{2n} A(c)$ .*

If the metric is changed into

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Re}(\operatorname{Tr}(A^* B)), \quad A, B \in \mathfrak{u}(k),$$

then the the result will be done into  $\theta = 2 \cdot \frac{1}{n} A(c)$ , and from  $e^{i\theta} I_n = e^{i\theta} I_n$ , it can be read as

$$i\theta I_n \in \mathfrak{u}(n) \quad \text{and} \quad \operatorname{Tr}(i\theta I_n) = i n \theta = 2i A(c),$$

where  $A(c)$  is the changed area induced from the changed metric on  $G_{n,m}$ .

On the other hand, Choi and Lee [2] thought of the dual version of Pinkall's result over  $\mathbb{C}H^m$  :

**Theorem 1.2.** [2] *Let  $S^1 \rightarrow S^{2m,1} \rightarrow \mathbb{C}H^m$  be the natural fibration. Let  $S$  be a complete totally geodesic surface in  $\mathbb{C}H^m$ , and  $\xi_S$  be the pullback bundle over  $S$ . Let  $c$  be a piecewise smooth, simple closed curve on  $S$ . Then the holonomy displacement along  $c$  is given by*

$$V(c) = e^{\frac{1}{2}A(c)i} \text{ or } e^{0i} \in S^1,$$

*where  $A(c)$ <sup>1</sup> is the area of the region on the surface  $S$  surrounded by  $c$ , depending on whether  $S$  is a complex submanifold or not.*

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<sup>1</sup> In [2], the bundle  $S^1 \rightarrow S^{2m,1} \rightarrow \mathbb{C}H^m$  is being studied through the one  $U(1) \rightarrow U(1,m)/U(m) \rightarrow U(1,m)/(U(1) \times U(m))$  with  $U(1,m)/U(m) \cong S^{2m,1}$  and  $U(1,m)/(U(1) \times U(m)) \cong \mathbb{C}H^m$ , and the unmentioned diffeomorphism  $f : \mathbb{C}H^m \rightarrow U(1,m)/(U(1) \times U(m))$  is a conformal map satisfying  $|f_*v| = \frac{1}{2}|v|$ .

Consider  $D_{n,m} = U(n, m)/(U(n) \times U(m))$ .

We generalize these results in the principal  $U(n)$ -bundle

$$U(n) \rightarrow U(n, m)/U(m) \xrightarrow{\pi} D_{n,m}$$

over  $D_{n,m}$  up to  $M_{m \times n}(\mathbb{C})$  for general positive integers  $n, m \in \mathbb{N}$ , not only up to  $U_{m,n}(\mathbb{C})$ , as follows: consider a left invariant metric on  $U(n, m)$ , related to the Killing-Cartan form, given by

$$(1-1) \quad \langle A, B \rangle = \frac{1}{2} \text{Re}(\text{Tr}(A^* B)), \quad A, B \in \mathfrak{u}(n, m),$$

and the induced metric on  $D_{n,m}$ , which makes the natural projection

$$\tilde{\pi} : U(n, m) \longrightarrow D_{n,m}$$

a Riemannian submersion and induces another Riemannian submersion

$$\pi : U(n, m)/U(m) \longrightarrow D_{n,m}.$$

Given a nontrivial  $X \in M_{m \times n}(\mathbb{C})$ , let

$$\widehat{X} = \begin{pmatrix} O_n & X^* \\ X & O_m \end{pmatrix} \quad \text{and} \quad i\widehat{X} = \begin{pmatrix} O_n & -iX^* \\ iX & O_m \end{pmatrix}.$$

Then for  $W = \frac{1}{|\widehat{X}|}X$ , for  $a, b \in \mathbb{R}$  and for  $z = a + bi \in \mathbb{C}$

$$z\widehat{W} = \frac{a+bi}{|\widehat{X}|}X = \frac{a}{|\widehat{X}|}\widehat{X} + \frac{b}{|i\widehat{X}|}i\widehat{X} = a\widehat{W} + bi\widehat{W}.$$

Consider a complete surface  $\tilde{S} = \{e^{z\widehat{W}}|z \in \mathbb{C}\}$  in  $U(n, m)$ . Then the map  $z \mapsto e^{z\widehat{W}} : \mathbb{C} \rightarrow \tilde{S}$  is a bijection. For  $g \in U(n, m)$ , define a complex surface  $S$  related to  $(X, g)$  in  $D_{n,m}$  by  $S = \tilde{\pi}(g\tilde{S})$ , which has a complex structure induced from a 2-dimensional subspace  $\mathfrak{m}' = \text{Span}_{\mathbb{R}}\{\widehat{X}, i\widehat{X}\} = \{z\widehat{W}|z = x + iy \text{ for } x, y \in \mathbb{R}\} \subset \mathfrak{m} \subset \mathfrak{u}(n, m)$ , where  $\mathfrak{m}$  is the orthogonal complement of  $\mathfrak{u}(n) + \mathfrak{u}(m)$ .

In case of

$$U(1) \rightarrow U(1, m)/U(m) \rightarrow U(1, m)/(U(1) \times U(m))$$

and in case of

$$U(n) \rightarrow U(n, 1)/U(1) \rightarrow U(n, 1)/(U(n) \times U(1)),$$

the Lie algebra generated by  $\{\widehat{X}, i\widehat{X}\}$  from  $X \in M_{m \times n}(\mathbb{C})$ , either  $m = 1$  or  $n = 1$ , inducing a complex surface  $S$  in the base space, produces a 3-dimensional Lie subgroup  $\widehat{G}$  of  $U(n, m)$ , which is isomorphic to  $SU(1, 1)$ , and a bundle structure isomorphic to

$$S(U(1) \times U(1)) \longrightarrow SU(1, 1) \longrightarrow SU(1, 1)/S(U(1) \times U(1)).$$

Then, under the notation  $A(c)$  of the induced area of the region surrounded by  $c$  on the surface  $S$  from the metric on the base space, this enables us to guess the pull-back bundle and the holonomy displacement<sup>2</sup>

$$V(c) = e^{2A(c)i}$$

along a curve  $c$  in the base space [2], which also enables us to guess its induced holonomy displacement in the original bundle from either  $X^*X \in \mathbb{R}$  or  $XX^* \in \mathbb{R}$ , from Lemma 2.2 and from Proposition 1.4. We deal with the latter one in Section 3.

But for general positive integers  $n, m \in \mathbb{N}$  and for a general nontrivial  $X \in M_{m \times n}(\mathbb{C})$ , the Lie algebra generated by  $\{\widehat{X}, i\widehat{X}\}$  is not 3-dimensional. Furthermore, the holonomy displacement depends not only on  $X$  but also on some 2-form of the complex surface  $S$  related to it too heavily. From now on, we consider two kinds of 2-forms on  $S$ , the first one is related to  $(X, g) \in M_{m \times n}(\mathbb{C}) \times U(n, m)$ , defined in Definition 1.3, and the other one is induced from the metric on  $S$  obtained from the metric on  $D_{n, m}$ , mentioned in Proposition 1.4.

To begin with, given  $X \in M_{m \times n}(\mathbb{C})$ , think of  $W = \frac{1}{|\widehat{X}|}X$  and  $\tilde{S} = \{e^{z\widehat{W}}|z \in \mathbb{C}\}$ , which is one to one correspondent to  $S_0 := \tilde{\pi}(\tilde{S})$ . Refer to Remark 1.11. And for any  $\tilde{g} \in U(n, m)$ , let  $\mathbb{L}_{\tilde{g}} : D_{n, m} \rightarrow D_{n, m}$  be the action of  $\tilde{g}$ , induced from the left multiplication of  $\tilde{g}$  on  $U(n, m)$ , which is an isometry from  $\mathbb{L}_{\tilde{g}} \circ \tilde{\pi} = \tilde{\pi} \circ L_{\tilde{g}}$ .

**Definition 1.3.** Think of a bundle  $U(n) \times U(m) \rightarrow U(n, m) \xrightarrow{\tilde{\pi}} D_{n, m}$ . Given  $(X, g) \in M_{m \times n}(\mathbb{C}) \times U(n, m)$ , consider  $W = \frac{1}{|\widehat{X}|}X$  and define a 2-form  $\omega_{(X, g)}$  on a complex surface  $S$  related to  $(X, g)$  in  $D_{n, m}$ , called an area form  $\omega_{(X, g)}$  related to  $(X, g)$  on  $S$ , by

$$\omega_{(X, g)}(\tilde{\pi}_*x, \tilde{\pi}_*y) = \det \begin{pmatrix} \langle L_{g_1^{-1}*}x, \widehat{W} \rangle & \langle L_{g_1^{-1}*}y, \widehat{W} \rangle \\ \langle L_{g_1^{-1}*}x, i\widehat{W} \rangle & \langle L_{g_1^{-1}*}y, i\widehat{W} \rangle \end{pmatrix}$$

under the identification of the tangent space of  $U(n, m)$  at the identity and its Lie algebra  $\mathfrak{u}(n, m)$ , where  $g_1 \in g\tilde{S} = \{ge^{z\widehat{W}}|z \in \mathbb{C}\}$  and both  $x$  and  $y$  are tangent to  $g\tilde{S}$  at  $g_1$ .

**Proposition 1.4.** Think of a bundle  $U(n) \times U(m) \rightarrow U(n, m) \xrightarrow{\tilde{\pi}} D_{n, m}$  such that  $\tilde{\pi}$  is a Riemannian submersion. Given a nontrivial  $X \in M_{m \times n}(\mathbb{C})$ , consider a complete, complex surface  $\tilde{S} = \{e^{z\widehat{W}}|z \in \mathbb{C}\}$  in  $U(n, m)$  for  $W = \frac{1}{|\widehat{X}|}X$  and for  $\widehat{W} = \frac{1}{|\widehat{X}|}X = \frac{1}{|\widehat{X}|}\widehat{X}$ .

(i) For a complex surface  $S_0 = \tilde{\pi}(\tilde{S})$  related to  $(X, e)$  in  $D_{n, m}$ , let  $\omega_0$

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<sup>2</sup> As explained in the footnote to  $A(c)$  in Theorem 1.2,  $V(c)$  in [2] is given by  $e^{\frac{1}{2}A(c)i}$ .

be the area form  $\omega_{(X,e)}$  related to  $(X,e)$  on  $S_0$ , where  $e$  is the identity of  $U(n,m)$ . Consider a coordinate system  $(r,\theta)$  on  $S_0$  induced from

$$S_0 = \{e^{\widehat{re^{i\theta}W}} \mid z = re^{i\theta} \in \mathbb{C}\}.$$

Then, for the differential  $d = d_{S_0}$  on  $S_0$ ,

$$\omega_0 = \omega_{(X,e)} = d\left(\sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r) d\theta\right),$$

where  $\sigma_1 \geq \dots \geq \sigma_n$  are the square roots of decreasingly ordered non-negative eigenvalues of  $W^*W$  with  $\sigma_1^2 + \dots + \sigma_n^2 = 1$ . (Refer to Lemma 2.2 for more information on the eigenvalues.)

(ii) Given a complex surface  $S$  in  $D_{n,m}$ , related to  $(X,g) \in M_{m \times n}(\mathbb{C}) \times U(n,m)$ , the area form  $\omega_{(X,g)}$  related to  $(X,g)$  on  $S$  is given by

$$\omega_{(X,g)} = \mathbb{L}_{g^{-1}}^* \omega_0.$$

(iii) The induced area form  $\omega$  on  $S$  from the metric on  $D_{n,m}$  is given by

$$\omega = \mathbb{L}_{g^{-1}}^* \omega_1$$

and

$$\omega_1 = \frac{1}{2} \sqrt{\sum_{j=1}^n \sinh^2(2\sigma_j r)} dr \wedge d\theta,$$

where  $\omega_1$  is the induced area form on  $S_0$  from the metric on  $D_{n,m}$ .

Especially, if the positive eigenvalues of  $W^*W$  consist of a single real number with allowing the duplication, that is,  $\sigma_1 = \dots = \sigma_q > 0 = \sigma_{q+1} = \dots = \sigma_n$ , then

$$\omega_1 = \omega_0 = d\left(\frac{q}{2} \sinh^2\left(\frac{r}{\sqrt{q}}\right) d\theta\right)$$

and  $q \in \mathbb{N}$  is the algebraic multiplicity of the positive eigenvalue  $\sigma_1^2$  of  $W^*W$ .

**Theorem 1.5.** Consider a bundle  $U(n) \rightarrow U(n,m)/U(m) \xrightarrow{\pi} D_{n,m}$  such that  $\pi$  is a Riemannian submersion. Given a complex surface  $S$  in  $D_{n,m}$ , related to  $(X,g) \in M_{m \times n}(\mathbb{C}) \times U(n,m)$ , let  $c$  be a smooth, simple, closed, orientation-preserving curve on  $S$ , parametrized by  $0 \leq t \leq 1$ , and  $\hat{c}$  its horizontal lift. Then the holonomy displacement  $\hat{c}(1) = \hat{c}(0) \cdot e^\Psi$  is given by the right action of  $e^\Psi$  for some  $\Psi \in \text{Span}_{\mathbb{R}}\{i(X^*X)^k\}_{k=1}^q \subset \mathfrak{u}(n)$ ,  $q = \text{rk}X$ , such that

$$\text{Tr}(\Psi) = 2i \text{Area}(c),$$

where  $\text{rk}X$  is the rank of  $X$  and  $\text{Area}(c)$  is the area of the region on the surface  $S$  surrounded by  $c$  with respect to the area form  $\omega_{(X,g)}$  related to  $(X,g)$  on  $S$ .

(For more information on how to find a concrete  $\Psi \in \mathfrak{u}(n)$  not only for  $n = 1$  but also for  $n > 1$ , refer to Remark 1.9 and 1.10.)

**Remark 1.6.** One metric structure on  $D_{n,m}$  induced from a Riemannian submersion  $\tilde{\pi} : U(n, m) \rightarrow D_{n,m}$  and the other one induced from a Riemannian submersion  $\pi : U(n, m)/U(m) \rightarrow D_{n,m}$ , in fact, are same. See Section 2.

**Corollary 1.7.** In addition to the hypothesis of Theorem 1.5, assume that  $X^*X$  has a single positive eigenvalue with algebraic multiplicity  $q$ . Then,

$$\begin{aligned}\Psi &= A \operatorname{diag} \left[ \underbrace{\frac{2i}{q} \operatorname{Area}(c), \dots, \frac{2i}{q} \operatorname{Area}(c)}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}} \right] A^* \\ &= A \operatorname{diag} \left[ \underbrace{\frac{2i}{q} A(c), \dots, \frac{2i}{q} A(c)}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}} \right] A^*\end{aligned}$$

for some  $A \in U(n)$ , where  $A(c)$  is the area with respect to the induced metric on  $S$  from the metric on  $D_{n,m}$ , i.e., the area with respect to the induced area form  $\omega$  on  $S$  from the metric on  $D_{n,m}$ .

Especially, if  $q = n$ , then  $X \in U_{m,n}(\mathbb{C})$ ,  $S$  is totally geodesic and  $e^\Psi = e^{i\theta} I_n$  with  $\theta = \frac{2}{n} \operatorname{Area}(c) = \frac{2}{n} A(c)$

*Proof.* From hypothesis, for  $W = \frac{1}{|X|} X$ ,  $W^*W$  also has a single positive eigenvalue with algebraic multiplicity  $q$ . Call its positive square root  $\sigma$ . Then, under the notation of Lemma 2.2, we get, for some  $A \in U(n)$ ,

$$\begin{aligned}W^*W &= A \Sigma^* \Sigma A^* \\ &= A \operatorname{diag} \left[ \underbrace{\sigma^2, \dots, \sigma^2}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}} \right] A^* \\ &= A \operatorname{diag} \left[ \underbrace{\frac{1}{q}, \dots, \frac{1}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}} \right] A^*\end{aligned}$$

from Proposition 1.4. Then, Theorem 1.5 and Proposition 1.4 say that

$$\Psi \in \operatorname{Span}\{X^*X\} = \operatorname{Span}\{W^*W\}$$

with

$$\operatorname{Tr}(\Psi) = 2i \operatorname{Area}(c) = 2i A(c),$$

which proves the former part of Corollary.

For the latter one, assume  $q = n$ . Then the similar arguments as those for  $W^*W = A \Sigma^* \Sigma A^*$  before show that

$$(1-2) \quad X^*X = \tilde{A}(\lambda I_n) \tilde{A} = \lambda I_n \in U_{m,n}(\mathbb{C})$$

for some  $\lambda > 0$  and for some  $\tilde{A} \in U(n)$ . And  $q = n$  implies

$$\Psi = A \left( \frac{2i}{n} \operatorname{Area}(c) I_n \right) A^* = \frac{2i}{n} \operatorname{Area}(c) I_n = i\theta I_n$$

for  $\theta = \frac{2}{n} \operatorname{Area}(c)$  and for some  $A \in U(n)$ , which gives

$$e^\Psi = e^{i\theta I_n} = e^{i\theta} I_n.$$

Furthermore, note that

$$W = \frac{1}{\sqrt{n\lambda}}X \quad \text{and} \quad W^*W = \frac{1}{n}I_n$$

from equation (1-2). Then for

$$\widehat{V} = \begin{pmatrix} iW^*W & O \\ O & -iWW^* \end{pmatrix} = \begin{pmatrix} \frac{i}{n}I_n & O \\ O & -iWW^* \end{pmatrix},$$

a set  $\{\widehat{X}, i\widehat{X}, \widehat{V}\}$ , or

$$\{\frac{1}{\sqrt{\lambda}}\widehat{X}, \frac{1}{\sqrt{\lambda}}i\widehat{X}, n\widehat{V}\} = \{\sqrt{n}\widehat{W}, \sqrt{n}i\widehat{W}, n\widehat{V}\}$$

generates a 3-dimensional Lie algebra with

$$[\sqrt{n}\widehat{W}, \sqrt{n}i\widehat{W}] = 2n\widehat{V}, \quad [n\widehat{V}, \sqrt{n}\widehat{W}] = -2\sqrt{n}i\widehat{W} \quad \text{and} \quad [n\widehat{V}, \sqrt{n}i\widehat{W}] = 2\sqrt{n}\widehat{W}.$$

Since each action of  $g \in U(n, m)$ ,  $\mathbb{L}_g : D_{n,m} \rightarrow D_{n,m}$  is an isometry from  $\mathbb{L}_g \circ \tilde{\pi} = \tilde{\pi} \circ L_g$  and  $S = \mathbb{L}_g(S_0)$ , Proposition 2.3 says that  $S$  is totally geodesic.  $\square$

**Remark 1.8.** For general  $q = 1, \dots, n$ , we can show that  $S$  in Corollary 1.7 is totally geodesic by using singular value decomposition: under the notation of Lemma 2.2, we get, for some  $B \in U(m)$ ,

$$\begin{aligned} WW^* &= B\Sigma\Sigma^*B^* \\ &= B \operatorname{diag}[\underbrace{\sigma^2, \dots, \sigma^2}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(m-q) \text{ times}}] B^* \\ &= B \operatorname{diag}[\underbrace{\frac{1}{q}, \dots, \frac{1}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(m-q) \text{ times}}] B^*. \end{aligned}$$

If we let

$$\Omega = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

and

$$\widehat{V} = \begin{pmatrix} iW^*W & O \\ O & -iWW^* \end{pmatrix},$$

then by using singular value decomposition, we get a set  $\{\widehat{X}, i\widehat{X}, \widehat{V}\}$ , or

$$\{\sqrt{q}\widehat{W}, \sqrt{q}i\widehat{W}, q\widehat{V}\}$$

generates a 3-dimensional Lie algebra with

$$[\sqrt{q}\widehat{W}, \sqrt{q}i\widehat{W}] = 2q\widehat{V}, \quad [q\widehat{V}, \sqrt{q}\widehat{W}] = -2\sqrt{q}i\widehat{W} \quad \text{and} \quad [q\widehat{V}, \sqrt{q}i\widehat{W}] = 2\sqrt{q}\widehat{W},$$

so Proposition 2.3 says that  $S$  is totally geodesic.

**Remark 1.9.**  $\Psi$  in Theorem 1.5 is obtained by a solution of a system of first order linear differential equations: for  $t \in [0, 1]$ , consider a curve  $z(t) = r(t)e^{i\theta(t)}$  in  $\mathbb{C}$  and another one  $\tilde{\gamma}(t)$  in  $\tilde{S} \subset U(n, m)$  such that  $\tilde{\gamma}(t) = e^{\widehat{z(t)W}}$  and that  $\tilde{\pi} \circ L_g \circ \tilde{\gamma} = c$ , where  $W = \frac{1}{|\hat{X}|}X$  and  $L_g : U(n, m) \rightarrow U(n, m)$  is an isometry by the left multiplication of  $g \in U(n, m)$ . For a curve  $\Psi(t) \in \mathfrak{u}(n)$ , given by

$$\Psi(t) = \begin{pmatrix} i \sum_{k=1}^q \phi_k(t)(W^*W)^k & O \\ O & O_m \end{pmatrix} \in \mathfrak{u}(n),$$

let  $\hat{c}(t) = (g\tilde{\gamma}(t)e^{\Psi(t)})U(m)$  be a horizontal lift of  $c(t)$ , where  $\phi_k(t)$  is to be determined for each  $k = 1, \dots, q$ . Then  $\Psi$  is given by  $\Psi = \Psi(1) - \Psi(0)$ . Furthermore, for some  $A \in U(n)$  and  $B \in U(m)$  from Lemma 2.2, we will get

$$\Psi(t) = \Omega \operatorname{diag} \left[ i \sum_{k=1}^q \sigma_1^{2k} \phi_k(t), \dots, i \sum_{k=1}^q \sigma_q^{2k} \phi_k(t), \underbrace{0, \dots, 0}_{(n-q) \text{ times}}, \underbrace{0, \dots, 0}_m \right] \Omega^*,$$

where

$$\Omega = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

and a system of a first order linear differential equations

$$(1-3) \quad \sum_{k=1}^q \sigma_j^{2k} \phi'_k(t) = \theta'(t) \sinh^2(\sigma_j r(t)), \quad j = 1, \dots, q.$$

Refer to Section 5 to see why this system comes. And for the concrete expression of  $\Psi(t)$ , refer to Remark 1.10.

**Remark 1.10.** For simplicity, we can regard  $\Psi(t)$  in Remark 1.9 as

$$\Psi(t) = i \sum_{k=1}^q \phi_k(t)(W^*W)^k \in \mathfrak{u}(n).$$

Then we get

$$\Psi(t) = A \operatorname{diag} \left[ i \sum_{k=1}^q \sigma_1^{2k} \phi_k(t), \dots, i \sum_{k=1}^q \sigma_q^{2k} \phi_k(t), \underbrace{0, \dots, 0}_{(n-q) \text{ times}} \right] A^*.$$

If  $\{\sigma_1, \dots, \sigma_q\} = \{\sigma_{j_1}, \dots, \sigma_{j_p}\}$  with  $\sigma_{j_1} > \dots > \sigma_{j_p}$ , then for uniquely determined  $(n \times n)$ -matrices  $I_{j_l}$ 's from the equation



$\text{diag}[\sigma_1^2, \dots, \sigma_q^2, 0, \dots, 0] = \sum_{l=1}^p \sigma_{j_l}^2 I_{j_l}$ , we have

$$\begin{aligned} W^*W &= A \text{diag}[\sigma_1^2, \dots, \sigma_q^2, \underbrace{0, \dots, 0}_{(n-q) \text{ times}}] A^* \\ &= A \left( \sum_{l=1}^p \sigma_{j_l}^2 I_{j_l} \right) A^* \\ &= \sum_{l=1}^p \sigma_{j_l}^2 A I_{j_l} A^*. \end{aligned}$$

Note

$$\begin{aligned} (W^*W)^k &= A \text{diag}[\sigma_1^{2k}, \dots, \sigma_q^{2k}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}}] A^* \\ &= A \left( \sum_{l=1}^p \sigma_{j_l}^{2k} I_{j_l} \right) A^* \\ &= \sum_{l=1}^p \sigma_{j_l}^{2k} A I_{j_l} A^* \end{aligned}$$

for  $k \in \{1, \dots, q\}$ , which says that  $\text{Span}_{\mathbb{R}}\{i(X^*X)^k\}_{k=1}^q$  in Theorem 1.5 has a dimension  $p$  and a basis  $\{A I_{j_l} A^*\}_{l=1}^p$ . For  $k = 1, \dots, p$ , this system can be regarded as

$$\begin{pmatrix} \sigma_{j_1}^{2 \cdot 1} & \cdots & \sigma_{j_p}^{2 \cdot 1} \\ \vdots & \ddots & \vdots \\ \sigma_{j_1}^{2p} & \cdots & \sigma_{j_p}^{2p} \end{pmatrix} \begin{pmatrix} A I_{j_1} A^* \\ \vdots \\ A I_{j_p} A^* \end{pmatrix} = \begin{pmatrix} (W^*W)^1 \\ \vdots \\ (W^*W)^p \end{pmatrix}.$$

Then for the invertible  $(p \times p)$ -matrix  $D$  on the left hand side,

$$D = \begin{pmatrix} \sigma_{j_1}^{2 \cdot 1} & \cdots & \sigma_{j_p}^{2 \cdot 1} \\ \vdots & \ddots & \vdots \\ \sigma_{j_1}^{2p} & \cdots & \sigma_{j_p}^{2p} \end{pmatrix},$$

we get

$$\begin{pmatrix} A I_{j_1} A^* \\ \vdots \\ A I_{j_p} A^* \end{pmatrix} = D^{-1} \begin{pmatrix} (W^*W)^1 \\ \vdots \\ (W^*W)^p \end{pmatrix},$$

which says that  $\{(W^*W)^k\}_{k=1}^p$  is also a basis. From this basis, the curve  $\Psi(t)$  may be reconstructed by

$$\begin{aligned}\Psi(t) &= \sum_{k=1}^p i\psi_k(t)(W^*W)^k \\ &= A \left( \sum_{l=1}^p \left( \sum_{k=1}^p i\psi_k(t)\sigma_{j_l}^{2k} \right) I_{j_l} \right) A^*,\end{aligned}$$

which change a system of first order differential equations (1–3) for constructing a horizontal curve condition into another one

$$\sum_{k=1}^p \sigma_{j_l}^{2k} \psi'_k(t) = \theta'(t) \sinh^2(\sigma_{j_l} r(t)), \quad l = 1, \dots, p,$$

whose initial value is given by  $\Psi(0)$  satisfying  $\hat{c}(0) = (g \tilde{\gamma}(0) e^{\Psi(0)}) U(m)$ . This system can be rewritten as

$$\begin{pmatrix} \sigma_{j_1}^{2 \cdot 1} & \dots & \sigma_{j_1}^{2p} \\ \vdots & \ddots & \vdots \\ \sigma_{j_p}^{2 \cdot 1} & \dots & \sigma_{j_p}^{2p} \end{pmatrix} \begin{pmatrix} \psi'_1(t) \\ \vdots \\ \psi'_p(t) \end{pmatrix} = \begin{pmatrix} \theta'(t) \sinh^2(\sigma_{j_1} r(t)) \\ \vdots \\ \theta'(t) \sinh^2(\sigma_{j_p} r(t)) \end{pmatrix}.$$

Then for the invertible  $(p \times p)$ -matrix  $C$  on the left hand side,

$$C = \begin{pmatrix} \sigma_{j_1}^{2 \cdot 1} & \dots & \sigma_{j_1}^{2p} \\ \vdots & \ddots & \vdots \\ \sigma_{j_p}^{2 \cdot 1} & \dots & \sigma_{j_p}^{2p} \end{pmatrix} = D^T,$$

we get a solution

$$\begin{pmatrix} \psi_1(t) \\ \vdots \\ \psi_p(t) \end{pmatrix} = \int C^{-1} \begin{pmatrix} \theta'(t) \sinh^2(\sigma_1 r(t)) \\ \vdots \\ \theta'(t) \sinh^2(\sigma_p r(t)) \end{pmatrix} dt.$$

Especially, if  $X \in U_{m,n}(\mathbb{C})$ , then  $p = 1$ ,  $\sigma_1 = \dots = \sigma_n$ ,

$$\text{Span}_{\mathbb{R}}\{i(X^*X)^k\}_{k=1}^q = \{i\mu I_{n \times n} \mid \mu \in \mathbb{R}\},$$

and

$$\psi_1(t) = \int \frac{1}{\sigma_1^2} \theta'(t) \sinh^2(\sigma_1 r(t)) dt,$$

so  $\Psi(t)$  is given by

$$\begin{aligned}\Psi(t) &= A(i\sigma_1^2 \psi_1(t) I_{n \times n}) A^* \\ &= i\sigma_1^2 \psi_1(t) I_{n \times n} \\ &= \Psi(0) + i \left( \int_0^t \theta'(u) \sinh^2(\sigma_1 r(u)) du \right) I_{n \times n}.\end{aligned}$$

**Remark 1.11.** *The singular value decomposition of  $e^{\widehat{zW}}$ , consisting of  $\Omega, \Gamma_n(z), \Gamma_m(z)$  and  $\Lambda(z)$ , in Section 4 shows that  $z \mapsto e^{\widehat{zW}} : \mathbb{C} \rightarrow \tilde{S}$  is a bijection. Furthermore, it also shows that the restriction of  $\tilde{\pi}$  on  $\tilde{S}$ ,*

$$\tilde{\pi} : \tilde{S} \rightarrow S_0$$

*is injective(, so bijective): to show it, assume that  $\tilde{\pi}(e^{\widehat{z_1 W}}) = \tilde{\pi}(e^{\widehat{z_2 W}})$  for  $z_j = r_j e^{i\theta_j}$ ,  $j = 1, 2$ , with  $r_j \geq 0$ . Then, we get*

$$e^{-\widehat{z_1 W}} e^{\widehat{z_2 W}} \in U(n) \times U(m)$$

*and the calculation through their singular value decomposition gives*

$$\begin{aligned} e^{-\widehat{z_1 W}} e^{\widehat{z_2 W}} &= \Omega \begin{pmatrix} \Gamma_n(z_1) & \Lambda(z_1)^* \\ \Lambda(z_1) & \Gamma_m(z_1) \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_n(z_2) & \Lambda(z_2)^* \\ \Lambda(z_2) & \Gamma_m(z_2) \end{pmatrix} \Omega^* \\ &= \Omega \begin{pmatrix} \Gamma_n(z_1) & -\Lambda(z_1)^* \\ -\Lambda(z_1) & \Gamma_m(z_1) \end{pmatrix} \begin{pmatrix} \Gamma_n(z_2) & \Lambda(z_2)^* \\ \Lambda(z_2) & \Gamma_m(z_2) \end{pmatrix} \Omega^*, \end{aligned}$$

*and so, from  $\Omega \in U(n) \times U(m)$ ,*

$$\begin{pmatrix} \Gamma_n(z_1) & -\Lambda(z_1)^* \\ -\Lambda(z_1) & \Gamma_m(z_1) \end{pmatrix} \begin{pmatrix} \Gamma_n(z_2) & \Lambda(z_2)^* \\ \Lambda(z_2) & \Gamma_m(z_2) \end{pmatrix} \in U(n) \times U(m),$$

*whose  $(1, n+1)$ -element is*

$$\begin{aligned} 0 &= \begin{pmatrix} \cosh(\sigma_1 r_1) & \underbrace{0 \cdots 0}_{(n-1) \text{ times}} & -e^{-i\theta_1} \sinh(\sigma_1 r_1) & \underbrace{0 \cdots 0}_{(m-1) \text{ times}} \end{pmatrix} \begin{pmatrix} e^{-i\theta_2} \sinh(\sigma_1 r_2) \\ 0 \\ \vdots \\ 0 \\ \cosh(\sigma_1 r_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= e^{-i\theta_2} \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) - e^{-i\theta_1} \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1) \\ &= (\cos \theta_2 \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) - \cos \theta_1 \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1)) \\ &\quad - i(\sin \theta_2 \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) - \sin \theta_1 \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1)). \end{aligned}$$

*Then*

$$(1-4) \quad \cos \theta_2 \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) = \cos \theta_1 \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1),$$

$$(1-5) \quad \sin \theta_2 \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) = \sin \theta_1 \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1),$$

*so from  $\cos^2 \theta + \sin^2 \theta = 1$ ,*

$$\cosh^2(\sigma_1 r_1) \sinh^2(\sigma_1 r_2) = \cosh^2(\sigma_1 r_2) \sinh^2(\sigma_1 r_1),$$

*which means either*

$$\begin{aligned} 0 &= \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) - \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1) \\ &= \sinh(\sigma_1(r_2 - r_1)) \end{aligned}$$

or

$$\begin{aligned} 0 &= \cosh(\sigma_1 r_1) \sinh(\sigma_1 r_2) + \cosh(\sigma_1 r_2) \sinh(\sigma_1 r_1) \\ &= \sinh(\sigma_1(r_2 + r_1)). \end{aligned}$$

Thus we get  $r_1 = r_2 \geq 0$ . If both of them equals to 0, then  $z_1 = z_2 = 0$ . If  $r_1 = r_2 > 0$ , then the equations (1-4) and (1-5) say that

$$\cos \theta_1 = \cos \theta_2 \quad \text{and} \quad \sin \theta_1 = \sin \theta_2,$$

which gives  $z_1 = r_1 e^{i\theta_1} = r_2 e^{i\theta_2} = z_2$ .

## 2. PRELIMINARIES

Given a submersion  $pr : (M, g_M) \rightarrow (B, g_B)$ , the *vertical distribution*  $\mathcal{V}$  and the *horizontal distribution*  $\mathcal{H} = \mathcal{V}^\perp$  are defined to be the kernel of  $pr_*$  and its orthogonal complement, respectively. And  $pr$  is said to be *Riemannian* if  $|pr_*x| = |x|$  for all  $x \in \mathcal{H}$ . [3]

Let  $K$  be a subgroup of the isometry group of a Riemannian manifold  $M$ , and suppose that all orbits have the same type, that is, any two are equivalantly diffeomorphic. Then there exists a differentiable structure on  $M/K$  with a Riemannian metric for each the natural projection  $\pi : M \rightarrow M/K$  is a Riemannian submersion. [3]

Given a Lie group  $G$  with a left-invariant metric and given a subgroup  $H$  with the right multiplication  $R_h$  an isometry for each  $h \in H$ , the space of left cosets  $G/H$  can be endowed with the metric for which the canonical projection  $\pi : G \rightarrow G/H$  is a Riemannian submersion. If we denote by  $\mathbb{L}_g : G/H \rightarrow G/H$  the action of  $g \in G$ , then  $\mathbb{L}_g \circ \pi = \pi \circ L_g$  and so  $\mathbb{L}_g$  is an isometry of  $G/H$ . [3]

Recall that

$$\begin{aligned} G &= U(n, m) = \{\Phi \in GL_{n+m}(\mathbb{C}) \mid \Phi^* \Lambda_m^n \Phi = \Lambda_m^n\} \\ &= \{\Phi \in GL_{n+m}(\mathbb{C}) \mid F(\Phi v, \Phi w) = F(v, w), v, w \in \mathbb{C}^{n+m}\} \end{aligned}$$

and that  $D_{n,m} := U(n, m) / (U(n) \times U(m))$  can be regarded as the set of  $n$ -dimensional subspaces  $V$  of  $\mathbb{C}^{n+m}$  such that  $F(v, v) \leq 0$  for every  $v \in V$  [5], where  $F : \mathbb{C}^{n+m} \rightarrow \mathbb{C}$  is an Hermitian form defined by

$$\begin{aligned} F(v, w) &= v^* \Lambda_m^n w \\ &= - \sum_{k=1}^n \bar{v}_k w_k + \sum_{s=n+1}^{n+m} \bar{v}_s w_s \end{aligned}$$

for column vectors  $v, w \in \mathbb{C}^{n+m}$ ,

$$\Lambda_m^n = \begin{pmatrix} -I_n & O_{n \times m} \\ O_{m \times n} & I_m \end{pmatrix}.$$

Consider the following canonical decomposition of the Lie algebra  $\mathfrak{u}(n, m)$  of  $G = U(n, m)$ :

$$\mathfrak{u}(n, m) = \mathfrak{h} + \mathfrak{m},$$

where

$$\mathfrak{h} = \mathfrak{u}(n) + \mathfrak{u}(m) = \left\{ \begin{pmatrix} A & O_{n \times m} \\ O_{m \times n} & B \end{pmatrix} : A \in \mathfrak{u}(n), B \in \mathfrak{u}(m) \right\}$$

and

$$\mathfrak{m} = \left\{ \hat{X} := \begin{pmatrix} O_n & X^* \\ X & O_m \end{pmatrix} : X \in M_{m \times n}(\mathbb{C}) \right\}.$$

Since the right multiplication  $R_h$ ,  $h \in U(n) \times U(m)$ , is an isometry with respect the left invariant metric given by the equation (1-1), there are two kinds of principal bundles

$$U(m) \rightarrow U(n, m) \xrightarrow{\hat{\pi}} U(n, m)/U(m)$$

and

$$U(n) \times U(m) \rightarrow U(n, m) \xrightarrow{\tilde{\pi}} D_{n,m}$$

such that both  $\hat{\pi}$  and  $\tilde{\pi}$  are Riemannian submersions. Note that each action of  $g \in U(n, m)$  on  $U(n, m)/U(m)$  and on  $D_{n,m}$ , denoted by

$$\hat{L}_g : U(n, m)/U(m) \longrightarrow U(n, m)/U(m),$$

and

$$\mathbb{L}_g : D_{n,m} \longrightarrow D_{n,m},$$

respectively, is an isometry from

$$\hat{L}_g \circ \hat{\pi} = \hat{\pi} \circ L_g \quad \text{and} \quad \mathbb{L}_g \circ \tilde{\pi} = \tilde{\pi} \circ L_g.$$

For each  $h_1 \in U(n)$ , consider the right action

$$\hat{R}_{h_1} : U(n, m)/U(m) \rightarrow U(n, m)/U(m)$$

given by

$$\hat{R}_{h_1}(gU(m)) = (gU(m)) \cdot h_1 := (gh)U(m),$$

where  $h = \begin{pmatrix} h_1 & O_{n \times m} \\ O_{m \times n} & I_m \end{pmatrix}$ . This action is well-defined from

$$\begin{pmatrix} h_1 & O \\ O & I_m \end{pmatrix} \begin{pmatrix} I_n & O \\ O & h_2 \end{pmatrix} = \begin{pmatrix} I_n & O \\ O & h_2 \end{pmatrix} \begin{pmatrix} h_1 & O \\ O & I_m \end{pmatrix}.$$

for any  $h_2 \in U(m)$ . By abusing of notations, write  $R_h = R_{h_1}$ . Then the equation  $\hat{R}_{h_1} \circ \hat{\pi} = \hat{\pi} \circ R_{h_1}$  implies that  $\hat{R}_{h_1} : U(n, m)/U(m) \rightarrow U(n, m)/U(m)$  is an isometry since  $R_{h_1} : U(n, m) \rightarrow U(n, m)$  is an isometry preserving each fiber of the bundle  $\hat{\pi} : U(n, m) \rightarrow U(n, m)/U(m)$ . More concretely, for any horizontal vector  $x$  with respect to the Riemannian submersion  $\hat{\pi} : U(n, m) \rightarrow U(n, m)/U(m)$ ,  $R_{h_1*} x$  is a horizontal vector since  $R_{h_1}$  is an

isometry preserving the fibers of the bundle  $\hat{\pi} : U(n, m) \rightarrow U(n, m)/U(m)$ , so

$$\begin{aligned} |\hat{R}_{h_1*} \hat{\pi}_* x| &= |\hat{\pi}_* R_{h_1*} x| \\ &= |R_{h_1*} x| \\ &= |x| \\ &= |\hat{\pi}_* x|. \end{aligned}$$

Consider another bundle

$$U(n) \rightarrow U(n, m)/U(m) \xrightarrow{\pi} D_{n, m}$$

and a metric structure on  $D_{n, m}$ , by regarding  $D_{n, m}$  as the space of orbits of  $U(n, m)/U(m)$  obtained from the action of a subgroup  $\{\hat{R}_{h_1} | h_1 \in U(n)\}$  of the isometry group of  $U(n, m)/U(m)$ , such that its projection  $\pi$  is a Riemannian submersion. In fact, for each  $g \in U(n, m)$ ,

$$g(U(n) \times U(m)) = \bigcup \{\hat{R}_{h_1}(gU(m)) \mid h_1 \in U(n)\},$$

which enables us to identify  $D_{n, m}$  with the space of orbits through

$$\pi(gU(m)) = g(U(n) \times U(m)),$$

and then we get  $\pi \circ \hat{\pi} = \tilde{\pi}$ .

For each  $g \in U(n, m)$  and for each  $h_1 \in U(n)$ , it is obvious that both  $\hat{L}_g$  and  $\hat{R}_{h_1}$  preserve the fibers of the bundle  $\pi : U(n, m)/U(m) \rightarrow D_{n, m}$ . In fact,  $\pi \circ \hat{R}_{h_1} = \pi$  by definition of  $\pi$  and  $\pi \circ \hat{L}_g = \mathbb{L}_g \circ \pi$  from

$$(\pi \circ \hat{L}_g) \circ \hat{\pi} = (\pi \circ \hat{\pi}) \circ L_g = \tilde{\pi} \circ L_g = \mathbb{L}_g \circ \tilde{\pi} = (\mathbb{L}_g \circ \pi) \circ \hat{\pi}.$$

Note that we have given two metric structures on  $D_{n, m}$ . In other words, we think of two Riemannian manifolds: the first one is the Riemannian manifold  $(D_{n, m}, \langle \cdot, \cdot \rangle_1)$  with  $\tilde{\pi} : U(n, m) \rightarrow (D_{n, m}, \langle \cdot, \cdot \rangle_1)$  a Riemannian submersion and the other one is the Riemannian manifold  $(D_{n, m}, \langle \cdot, \cdot \rangle_2)$  with  $\pi : U(n, m)/U(m) \rightarrow (D_{n, m}, \langle \cdot, \cdot \rangle_2)$  a Riemannian submersion. But,

$$\tilde{\pi} = \pi \circ \hat{\pi} \quad \text{and} \quad \hat{\pi} \circ L_{g^{-1}} = \hat{L}_{g^{-1}} \circ \hat{\pi}, \quad \forall g \in U(n, m)$$

say that these two metric structures are same, that is,

$$|\tilde{\pi}_* x|_1 = |\tilde{\pi}_* x|_2$$

for any horizontal vector  $x$  with respect to the Riemannian submersion  $\tilde{\pi} : U(n, m) \rightarrow (D_{n, m}, \langle \cdot, \cdot \rangle_1)$ . To show it, assume  $x \in T_g U(n, m)$ ,  $g \in U(n, m)$ . The identification of the tangent space of  $U(n, m)$  at the Identity and  $\mathfrak{u}(n, m)$  gives  $L_{g^{-1}*} x \perp (\mathfrak{u}(n) + \mathfrak{u}(m))$  and so  $L_{g^{-1}*} x \perp \mathfrak{u}(m)$ , which means that  $L_{g^{-1}*} x$  is horizontal with respect to  $\hat{\pi} : U(n, m) \rightarrow U(n, m)/U(m)$ . Then, since  $\hat{L}_{g^{-1}} : U(n, m)/U(m) \rightarrow U(n, m)/U(m)$  is an

isometry,  $\hat{\pi} \circ L_{g^{-1}} = \hat{L}_{g^{-1}} \circ \hat{\pi}$  says that  $x$  is also horizontal with respect to  $\hat{\pi} : U(n, m) \rightarrow U(n, m)/U(m)$ , in other words,  $|x| = |\hat{\pi}_* x|$ . More precisely,

$$|x| = |L_{g^{-1}*} x| = |\hat{\pi}_* L_{g^{-1}*} x| = |\hat{L}_{g^{-1}*} \hat{\pi}_* x| = |\hat{\pi}_* x|.$$

Furthermore,  $L_{g^{-1}*} x \perp (\mathfrak{u}(n) + \mathfrak{u}(m))$  also says that  $L_{g^{-1}*} x \perp \mathfrak{u}(n)$  and  $L_{g^{-1}*} x \perp \mathfrak{u}(m)$  at the same time, which means  $\hat{\pi}_* L_{g^{-1}*} x = L_{g^{-1}*} x + \mathfrak{u}(m)$  is perpendicular to  $\{V + \mathfrak{u}(m) \mid V \in \mathfrak{u}(n)\}$  at the origin of  $U(n, m)/U(m)$  from  $\mathfrak{u}(n) \perp \mathfrak{u}(m)$  and so horizontal with respect to  $\pi : U(n, m)/U(m) \rightarrow (D_{n,m}, \langle \cdot, \cdot \rangle_2)$  because  $\{V + \mathfrak{u}(m) \mid V \in \mathfrak{u}(n)\}$  is the kernel of  $\pi_*$  of the bundle  $\pi : U(n, m)/U(m) \rightarrow (D_{n,m}, \langle \cdot, \cdot \rangle_2)$  at the origin of  $U(n, m)/U(m)$  from  $[\mathfrak{u}(n), \mathfrak{u}(m)] = 0$ . Then, since  $\hat{L}_g : U(n, m)/U(m) \rightarrow U(n, m)/U(m)$  is an isometry preserving the fibers of the bundle  $\pi : U(n, m)/U(m) \rightarrow (D_{n,m}, \langle \cdot, \cdot \rangle_2)$ ,  $\hat{L}_{g*} \hat{\pi}_* L_{g^{-1}*} x$  is also horizontal with respect to  $\pi : U(n, m)/U(m) \rightarrow (D_{n,m}, \langle \cdot, \cdot \rangle_2)$ . And from

$$\hat{L}_{g*} \hat{\pi}_* L_{g^{-1}*} x = \hat{\pi}_* L_{g*} L_{g^{-1}*} x = \hat{\pi}_* x,$$

we get that  $\hat{\pi}_* x$  is horizontal with respect to  $\pi : U(n, m)/U(m) \rightarrow (D_{n,m}, \langle \cdot, \cdot \rangle_2)$  and that

$$|\hat{\pi}_* x| = |\pi_* \hat{\pi}_* x|_2,$$

Therefore,  $\tilde{\pi} = \pi \circ \hat{\pi}$  gives

$$|\tilde{\pi}_* x|_1 = |x| = |\hat{\pi}_* x| = |\pi_* \hat{\pi}_* x|_2 = |\tilde{\pi}_* x|_2.$$

Thus, we will not distinguish one metric on  $D_{n,m}$  from the other one.

**Lemma 2.1.** *Given a nontrivial  $X \in M_{m \times n}(\mathbb{C})$ , the Lie subalgebra of  $\mathfrak{u}(n, m)$ , generated by*

$$\hat{X} = \begin{pmatrix} O_n & X^* \\ X & O_m \end{pmatrix}, \quad i\hat{X} = \begin{pmatrix} O_n & -iX^* \\ iX & O_m \end{pmatrix},$$

is

$$\text{Span}_{\mathbb{R}}\{\tilde{V}_k, \tilde{X}_k, i\tilde{X}_k \mid k = 1, 2, \dots\},$$

where

$$\tilde{V}_k = \begin{pmatrix} i(X^* X)^k & O \\ O & -i(XX^*)^k \end{pmatrix},$$

$$\tilde{X}_k = \begin{pmatrix} O_n & (X^* X)^{k-1} X^* \\ X(X^* X)^{k-1} & O_m \end{pmatrix},$$

and

$$i\tilde{X}_k = \begin{pmatrix} O_n & -i(X^* X)^{k-1} X^* \\ iX(X^* X)^{k-1} & O_m \end{pmatrix}.$$

Furthermore, for  $q = \text{rk} X$ ,

$$(2-1) \quad \text{Span}_{\mathbb{R}}\{\tilde{V}_k \mid k = 1, \dots\} = \text{Span}_{\mathbb{R}}\{\tilde{V}_k \mid k = 1, \dots, q\},$$

is, at most, a  $q$ -dimensional subalgebra of  $\mathfrak{u}(n) + \mathfrak{u}(m)$ .

*Proof.* The first assertion can be given by direct calculations.  
For the second one, consider a bundle

$$U(n) \times U(m) \rightarrow U(n, m) \xrightarrow{\tilde{\pi}} D_{n, m}.$$

For  $X \in M_{m \times n}(\mathbb{C})$ , the following Lemma 2.2 make us consider three matrices  $A \in U(n)$ ,  $B \in U(m)$  and  $\Sigma \in M_{m \times n}(\mathbb{C})$  such that  $X = B\Sigma A^*$ . Then

$$X^*X = A\Sigma^*\Sigma A^*, \quad XX^* = B\Sigma\Sigma^*B^*$$

and

$$\tilde{V}_k = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} i(\Sigma^*\Sigma)^k & O \\ O & -i(\Sigma\Sigma^*)^k \end{pmatrix} \begin{pmatrix} A^* & O \\ O & B^* \end{pmatrix},$$

where

$$\Sigma^*\Sigma = \text{diag}[\sigma_1^2, \dots, \sigma_q^2, \underbrace{0, \dots, 0}_{(n-q) \text{ times}}] \in M_{n \times n}$$

and

$$\Sigma\Sigma^* = \text{diag}[\sigma_1^2, \dots, \sigma_q^2, \underbrace{0, \dots, 0}_{(m-q) \text{ times}}] \in M_{m \times m}$$

for  $\sigma_1 \geq \dots \geq \sigma_q > 0$ , the positive square roots of the decreasingly ordered nonzero eigenvalues of  $WW^*$ , which are the same as the decreasingly ordered nonzero eigenvalues of  $W^*W$ . Thus, the equation (2-1) is trivially obtained and so it is obvious that its dimension is less than or equal to  $q$ . And it is a Lie algebra from  $[\tilde{V}_k, \tilde{V}_j] = 0$  for  $k, j = 1, 2, \dots$ .  $\square$

The following Lemma on *Singular value decomposition* plays an important role in this paper.

**Lemma 2.2.** [4] *Let  $W \in M_{m \times n}(\mathbb{C})$  be given, put  $a = \min\{m, n\}$ , and suppose that  $\text{rk}W = q$ .*

(i) *There are unitary matrices  $A \in U(n)$  and  $B \in U(m)$  and a square diagonal matrix*

$$\Sigma_a = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_a \end{pmatrix}$$

*such that  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q > 0 = \sigma_{q+1} = \dots = \sigma_a$  and  $W = B\Sigma A^*$ , in which*

$$\Sigma = \begin{cases} \Sigma_a & \text{in case of } n = m, \\ (\Sigma_a \ O)^T \in M_{m \times n} & \text{in case of } n < m, \\ (\Sigma_a \ O) \in M_{m \times n} & \text{in case of } n > m. \end{cases}$$

(ii) *The parameters  $\sigma_1, \dots, \sigma_q$  are the positive square roots of the decreasingly ordered nonzero eigenvalues of  $WW^*$ , which are the same as the decreasingly ordered nonzero eigenvalues of  $W^*W$ .*



Recall the following proposition, which gives a sufficient condition to determine whether a given complex surface  $S$  in  $D_{n,m}$  related to  $(X, g)$  is totally geodesic or not.

**Proposition 2.3.** [5] *Let  $(G, H, \sigma)$  be a symmetric space and  $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$  the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces  $\mathfrak{m}'$  of  $\mathfrak{m}$  such that  $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$  and the set of complete totally geodesic submanifolds  $M'$  through the origin 0 of the affine symmetric space  $M = G/H$ , the correspondence being given by  $\mathfrak{m}' = T_0(M')$ .*

### 3. HOLONOMY DISPLACEMENT IN THE BUNDLE

$$U(n) \rightarrow U(n, 1)/U(n) \rightarrow U(n, 1)/(U(n) \times U(1))$$

Even though the result in this section can be obtained in view of Corollary 1.7, we deal with this section in the way which will be used in Section 5.

Given  $X \in M_{1 \times n}(\mathbb{C}) \cong \mathbb{C}^n$ , consider

$$\text{Span}_{\mathbb{R}}\{\widehat{X}, i\widehat{X}\} = \mathfrak{m}' \subset \mathfrak{m} \subset \mathfrak{u}(n, m).$$

Then, for  $\lambda = \sqrt{XX^*} = |\widehat{X}|$  and for  $W = \frac{1}{\lambda}X$ ,

$$\begin{aligned} \widehat{W} &= \frac{1}{\lambda}\widehat{X} = \begin{pmatrix} O_n & W^* \\ W & 0 \end{pmatrix}, \\ i\widehat{W} &= \frac{1}{\lambda}i\widehat{X} = \begin{pmatrix} O_n & -iW^* \\ iW & 0 \end{pmatrix}, \\ \widehat{V} &:= \begin{pmatrix} iW^*W & O \\ O & -i \end{pmatrix}, \end{aligned}$$

which generate a 3-dimensional Lie algebra  $\widehat{\mathfrak{g}}$  with  $\widehat{G}$  its Lie group such that

$$(3-1) \quad [\widehat{W}, i\widehat{W}] = 2\widehat{V}, \quad [\widehat{V}, \widehat{W}] = -2i\widehat{W}, \quad [\widehat{V}, i\widehat{W}] = 2\widehat{W}.$$

Since  $WW^* = 1$ , Lemma 2.2 says that, for  $\Sigma = (1 \ 0 \cdots 0) \in M_{1 \times n}(\mathbb{C})$ , there are  $A \in U(n)$  and  $\mu \in \mathbb{R}$  such that

$$W = B\Sigma A^* \in M_{1 \times n}(\mathbb{C}),$$

where  $B = (e^{i\mu}) \in U(1)$ , and then for

$$\Omega = \left( \begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & A & & 0 \\ \hline 0 & \cdots & 0 & e^{i\mu} \end{array} \right),$$

we get another expressios for  $\widehat{W}, i\widehat{W}, \widehat{V}$  through  $\text{Ad}_\Omega : \mathfrak{u}(n, m) \rightarrow \mathfrak{u}(n, m)$ ,

$$(3-2) \quad \widehat{W} = \Omega \left( \begin{array}{cccc|c} & & & & 1 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline & & O_n & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array} \right) \Omega^{-1}$$

$$(3-3) \quad i\widehat{W} = \Omega \left( \begin{array}{cccc|c} & & & & -i \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline & & O_n & & 0 \\ \hline i & 0 & \cdots & 0 & 0 \end{array} \right) \Omega^{-1}$$

and

$$(3-4) \quad \widehat{V} = \Omega \left( \begin{array}{cccc|c} i & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & -i \end{array} \right) \Omega^{-1}.$$

In fact,  $WW^* = 1$  implies that the nonzero eigenvalue of  $W^*W$  consists of a simple 1 from Lemma 2.2. More concretely, note that

$$\widehat{V} = \begin{pmatrix} iW^*W & O \\ O & -iWW^* \end{pmatrix} = \begin{pmatrix} iW^*W & O \\ O & -i \end{pmatrix}$$

and

$$W^*W = A\Sigma^*\Sigma A^* = A \begin{pmatrix} 1 & O \\ O & O_{n-1} \end{pmatrix} A^*.$$

Consider a complex surface  $S_0$  related to  $(X, e)$  in  $D_{n,1}$ , which is totally geodesic from Equation (3-1) and from Proposition 2.3, where  $e$  is the identity of  $U(n, m)$ . For a smooth, simple, closed, orientaion-preserving curve  $c : [0, 1] \rightarrow S_0$ , assume that  $\hat{c} : [0, 1] \rightarrow U(n, 1)/U(1)$ , one of its horizontal lifts, is given. Let  $\text{Area}(c)$  denote the area of the region on the surface  $S_0$  surrounded by  $c$  with respect to the area form  $\omega_0 = \omega_{(X, e)}$  related to  $(X, e)$  on  $S_0$ . Put  $A(c)$  denote the area with respect to the induced metric on  $S_0$  from the metric on  $D_{n,m}$ , *i.e.*, the area with respect to the induced area form  $\omega_1$  on  $S_0$  from the metric on  $D_{n,m}$ . Note that  $\omega_0 = \omega_1$  from Proposition 1.4.

*Claim)* There exists an element  $\Psi \in \mathfrak{u}(n)$  such that

$$\hat{c}(1) = \hat{c}(0) \cdot e^\Psi \quad \text{and} \quad \text{Tr}(\Psi) = 2i\text{Area}(c) = 2iA(c).$$

Consider a curve  $z(t) = r(t)e^{i\theta(t)}$  in  $\mathbb{C}$  and another one  $\tilde{\gamma}(t)$  in  $\tilde{S} = \{e^{\widehat{zW}} | z \in \mathbb{C}\} \subset U(n, 1)$  such that  $\tilde{\gamma}(t) = e^{\widehat{z(t)W}}$  is a lifting of  $c$ . Then for a horizontal lifting of  $\hat{c}$ , under the identification of  $\mathfrak{u}(n)$  and  $\mathfrak{u}(n) + \{0\}$ , we

can find a curve

$$\begin{aligned}\Psi(t) &= i\phi(t)W^*W \\ &= \begin{pmatrix} i\phi(t)W^*W & O \\ O & 0 \end{pmatrix} \\ &= \Omega \left( \begin{array}{cccc|c} i\phi(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 \end{array} \right) \Omega^{-1} \in \mathfrak{u}(n)\end{aligned}$$

such that

$$\hat{c}(t) = (\tilde{\gamma}(t)U(1)) \cdot e^{\Psi(t)} = (\tilde{\gamma}(t)e^{\Psi(t)})U(1).$$

Note that from

$$\widehat{z(t)W} = \Omega \left( \begin{array}{cccc|c} & & & & r(t)e^{-i\theta(t)} \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline r(t)e^{i\theta(t)} & 0 & \cdots & 0 & 0 \end{array} \right) \Omega^{-1},$$

we get

$$\tilde{\gamma}(t) = \Omega \left( \begin{array}{cccc|c} \cosh(r(t)) & 0 & \cdots & 0 & e^{-i\theta(t)} \sinh(r(t)) \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & I_{n-1} & & & 0 \\ \hline e^{i\theta(t)} \sinh(r(t)) & 0 & \cdots & 0 & \cosh(r(t)) \end{array} \right) \Omega^{-1}.$$

Then for  $\bar{c}(t) = \tilde{\gamma}(t)e^{\Psi(t)}$ ,

$$\bar{c}(t) = \Omega \left( \begin{array}{cccc|c} e^{i\phi(t)} \cosh(r(t)) & 0 & \cdots & 0 & e^{-i\theta(t)} \sinh(r(t)) \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & I_{n-1} & & & 0 \\ \hline e^{i(\theta(t)+\phi(t))} \sinh(r(t)) & 0 & \cdots & 0 & \cosh(r(t)) \end{array} \right) \Omega^{-1}$$

and from

$$\begin{aligned}L_{\bar{c}(t)^{-1} *} \dot{\bar{c}}(t) + \mathfrak{u}(1) \\ &= e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} (\tilde{\gamma}'(t)e^{\Psi(t)} + \tilde{\gamma}(t)e^{\Psi(t)} \Psi'(t)) + \mathfrak{u}(1) \\ &= (e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} \tilde{\gamma}'(t)e^{\Psi(t)} + \Psi'(t)) + \mathfrak{u}(1),\end{aligned}$$

$\hat{c}(t) = \bar{c}(t)U(1)$  is horizontal if and only if the first  $(n \times n)$ -block of  $e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} \tilde{\gamma}'(t)e^{\Psi(t)} + \Psi'(t)$  is a zero matrix, in other words,

$$i(\phi'(t) - \theta'(t) \sinh^2(r(t))) = 0, \quad i.e., \quad \phi'(t) = \theta'(t) \sinh^2(r(t)).$$

So, for  $\Psi := \Psi(1) - \Psi(0) \in \text{Span}_{\mathbb{R}}\{i(X^*X)\} \in \mathfrak{u}(n)$  and for the region  $D(\subset S)$  enclosed by the given orientation curve  $c$ , Proposition 1.4 says that

$$\begin{aligned}
\text{Tr}(\Psi) &= i(\phi(1) - \phi(0)) \\
&= i \int_0^1 \theta'(t) \sinh^2(r(t)) dt \\
&= 2i \int_{[0,1]} \frac{1}{2} \sinh^2(r(t)) \theta'(t) dt \\
&= 2i \int_c \frac{1}{2} \sinh^2 r d\theta \\
&= 2i \int_D d\left(\frac{1}{2} \sinh^2 r d\theta\right) \\
&= 2i \int_D \omega_0 \\
&= 2i \text{Area}(c) \\
&= 2i A(c).
\end{aligned}$$

Since  $\tilde{\gamma}(0) = \tilde{\gamma}(1)$  and  $[\Psi(0), \Psi(1)] = O_n$  in  $\mathfrak{u}(n)$ , we get

$$\bar{c}(0)^{-1} \bar{c}(1) = (\tilde{\gamma}(0) e^{\Psi(0)})^{-1} (\tilde{\gamma}(1) e^{\Psi(1)}) = e^{-\Psi(0)} e^{\Psi(1)} = e^{\Psi}$$

and so

$$\hat{c}(1) = \bar{c}(1)K = (\bar{c}(0) e^{\Psi}) K = \bar{c}(0)K \cdot e^{\Psi} = \hat{c}(0) \cdot e^{\Psi},$$

i.e., the holonomy displacement is given by the right action of  $e^{\Psi} \in U(n)$ .

#### 4. PROOF OF PROPOSITION 1.4

*Proof.* For the part (i), let

$$W = \frac{1}{|\hat{X}|} X, \quad \widehat{W} = \frac{1}{|\hat{X}|} \widehat{X} = \frac{1}{|\hat{X}|} \hat{X}, \quad \text{and} \quad i\widehat{W} = \frac{1}{|\hat{X}|} i\hat{X} = \frac{1}{|\hat{X}|} i\hat{X}$$

and consider an ordered pair  $(\widehat{W}, i\widehat{W})$ , which induces an oriented orthonormal basis of each tangent space of  $S_0$  and an area form  $\omega_0 = \omega_{(X,e)}$  related to  $(X, e)$  on  $S_0$ , where  $e$  is the identity of  $U(n, m)$ .

Assume  $\text{rk} W = q$ ,  $\min\{n, m\} = a$  and  $\text{Max}\{n, m\} = b$ . Then, Lemma 2.2 says that there are nonnegative real numbers

$$\sigma_1 \geq \cdots \geq \sigma_q > 0 = \sigma_{q+1} = \cdots = \sigma_a = \cdots = \sigma_b = \cdots = \sigma_{n+m}$$

such that  $\sigma_1, \dots, \sigma_q$  are positive square roots of nonzero eigenvalues of  $WW^*$  and  $W^*W$  and that  $\Sigma_a = \text{diag}[\sigma_1, \dots, \sigma_a]$  and

$$W = B\Sigma A^*, \quad A \in U(n), \quad B \in U(m),$$

where

$$\Sigma = \begin{cases} \Sigma_a & \text{in case of } n = m, \\ (\Sigma_a \ O)^T \in M_{m \times n} & \text{in case of } n < m, \\ (\Sigma_a \ O) \in M_{m \times n} & \text{in case of } n > m. \end{cases}$$

For  $r \geq 0$  and for  $\theta \in \mathbb{R}$ , let  $z = re^{i\theta}$  and consider  $\tilde{S} = \{e^{z\widehat{W}} \mid z \in \mathbb{C}\}$ . Let

$$\Omega = \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Then,

$$\begin{aligned} e^{z\widehat{W}} &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} (W^*W)^k & e^{-i\theta} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} r^{2k+1} (W^*W)^k W^* \\ e^{i\theta} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} r^{2k+1} W (W^*W)^k & \sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} (WW^*)^k \end{pmatrix} \\ &= \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \Gamma_n(z) & \Lambda(z)^* \\ \Lambda(z) & \Gamma_m(z) \end{pmatrix} \begin{pmatrix} A^* & O \\ O & B^* \end{pmatrix} \\ &= \Omega \begin{pmatrix} \Gamma_n(z) & \Lambda(z)^* \\ \Lambda(z) & \Gamma_m(z) \end{pmatrix} \Omega^* \end{aligned}$$

from

$$W(W^*W)^k = (WW^*)^k W, \quad W^*W = A\Sigma^*\Sigma A^* \quad \text{and} \quad WW^* = B\Sigma\Sigma^*B^*,$$

where

$$\Gamma_n(z) = \text{diag}[\cosh(\sigma_1 r), \dots, \cosh(\sigma_n r)],$$

$$\Gamma_m(z) = \text{diag}[\cosh(\sigma_1 r), \dots, \cosh(\sigma_m r)],$$

and for  $\Lambda_a(z) = \text{diag}[e^{i\theta} \sinh(\sigma_1 r), \dots, e^{i\theta} \sinh(\sigma_a r)]$ ,

$$\Lambda(z) = \begin{cases} \Lambda_a(z) & \text{in case of } a = n = m, \\ (\Lambda_a(z) \ O)^T \in M_{m \times n} & \text{in case of } a = n < m, \\ (\Lambda_a(z) \ O) \in M_{m \times n} & \text{in case of } a = m < n. \end{cases}$$

Note  $z \mapsto e^{z\widehat{W}} : \mathbb{C} \rightarrow \tilde{S}$  is a bijection.

By abusing notations, we use the same letter  $(r, \theta)$  on  $S_0$  for the induced one from the coordinate chart  $(r, \theta)$  on  $\tilde{S}$ . Then, the relation

$$(4-1) \quad \tilde{\pi}_* \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \circ \tilde{\pi} \quad \text{and} \quad \tilde{\pi}_* \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \circ \tilde{\pi}$$

holds. Note that, under the identification of the Lie algebra  $\mathfrak{u}(n, m)$  and  $T_e G$ , where  $G = U(n, m)$ , the direct calculation shows that

$$(4-2) \quad L_{e^{-z\widehat{W}} *} \frac{\partial}{\partial r} \Big|_{e^{z\widehat{W}}} = \begin{pmatrix} O & e^{-i\theta} W^* \\ e^{i\theta} W & O \end{pmatrix},$$

and

$$(4-3)$$

$$\begin{aligned} &L_{e^{-z\widehat{W}} *} \frac{\partial}{\partial \theta} \Big|_{e^{z\widehat{W}}} \\ &= \begin{pmatrix} \frac{-i}{2} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (2r)^{2k} (W^*W)^k & \frac{-i}{2} e^{-i\theta} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (2r)^{2k+1} (W^*W)^k W^* \\ \frac{i}{2} e^{i\theta} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (2r)^{2k+1} W (W^*W)^k & \frac{i}{2} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (2r)^{2k} (WW^*)^k \end{pmatrix}, \end{aligned}$$

so for  $W^*W = A\Sigma^*\Sigma A^* = A\Sigma_n^*\Sigma_n A^*$ , where  $\Sigma_n = \text{diag}[\sigma_1, \dots, \sigma_n]$ , we get

$$\begin{aligned}
& \omega_0 \left( \frac{\partial}{\partial r} \Big|_{\tilde{\pi}(e^{z\widehat{W}})}, \frac{\partial}{\partial \theta} \Big|_{\tilde{\pi}(e^{z\widehat{W}})} \right) \\
&= \det \begin{pmatrix} \langle L_{e^{-z\widehat{W}}} \frac{\partial}{\partial r} \Big|_{e^{z\widehat{W}}}, \widehat{W} \rangle & \langle L_{e^{-z\widehat{W}}} \frac{\partial}{\partial \theta} \Big|_{e^{z\widehat{W}}}, \widehat{W} \rangle \\ \langle L_{e^{-z\widehat{W}}} \frac{\partial}{\partial r} \Big|_{e^{z\widehat{W}}}, i\widehat{W} \rangle & \langle L_{e^{-z\widehat{W}}} \frac{\partial}{\partial \theta} \Big|_{e^{z\widehat{W}}}, i\widehat{W} \rangle \end{pmatrix} \\
&= \det \begin{pmatrix} \cos \theta & -\frac{1}{2} \sin \theta \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (2r)^{2k+1} \text{Tr}(W^*W)^{k+1} \\ \sin \theta & \frac{1}{2} \cos \theta \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (2r)^{2k+1} \text{Tr}(W^*W)^{k+1} \end{pmatrix} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (2r)^{2k+1} (\sigma_1^{2(k+1)} + \dots + \sigma_n^{2(k+1)}) \\
&= \frac{1}{2} \sum_{j=1}^n \sigma_j \sinh(2\sigma_j r) \\
&= \sum_{j=1}^n \sigma_j \sinh(\sigma_j r) \cosh(\sigma_j r).
\end{aligned}$$

Therefore,

$$\omega_0 = \sum_{j=1}^n \sigma_j \cosh(\sigma_j r) \sinh(\sigma_j r) dr \wedge d\theta = d \left( \sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r) d\theta \right).$$

Furthermore,

$$\widehat{W}^* \widehat{W} = \begin{pmatrix} W^*W & O \\ O & WW^* \end{pmatrix}$$

shows that

$$1 = |\widehat{W}|^2 = \text{Tr}(W^*W) = \text{Tr}(\Sigma_n^*\Sigma_n) = \sigma_1^2 + \dots + \sigma_n^2.$$

Before proving the part (ii), note that the restriction of  $\tilde{\pi}$  on  $\tilde{S}$  is a bijection to  $S_0$  from Remark 1.11, which induces that the restriction of  $\mathbb{L}_g$  on  $S_0$  is a bijection to  $S$  from  $\tilde{\pi} \circ L_g = \mathbb{L}_g \circ \tilde{\pi}$ . So, for  $\mathbb{L}_{g^{-1}}$ , its restriction  $\mathbb{L}_{g^{-1}} : S \rightarrow S_0$  is also a bijection.

To prove the part (ii), let  $(r_1, \theta_1)$  be a coordinate chart on  $S = \tilde{\pi}(g\tilde{S})$  with a complex structure induced from  $\tilde{S}$ , *i.e.*,

$$r_1(\tilde{\pi}(ge^{z\widehat{W}})) = r(e^{z\widehat{W}}) \quad \text{and} \quad \theta_1(\tilde{\pi}(ge^{z\widehat{W}})) = \theta(e^{z\widehat{W}}),$$

which says

$$r_1 \circ \tilde{\pi} \circ L_g = r \quad \text{and} \quad \theta_1 \circ \tilde{\pi} \circ L_g = \theta.$$

Then, from the equations (4-1),

$$\begin{aligned}
\frac{\partial}{\partial r_1} \Big|_{\tilde{\pi} \circ L_g(e^{z\widehat{W}})} &= (\tilde{\pi} \circ L_g)_* \frac{\partial}{\partial r} \Big|_{e^{z\widehat{W}}} \\
&= (\mathbb{L}_g \circ \tilde{\pi})_* \frac{\partial}{\partial r} \Big|_{e^{z\widehat{W}}} \\
&= \mathbb{L}_g^* \frac{\partial}{\partial r} \Big|_{\tilde{\pi}(e^{z\widehat{W}})},
\end{aligned}$$

so we get

$$\frac{\partial}{\partial r} \Big|_{\tilde{\pi}(e^z \widehat{W})} = \mathbb{L}_{g^{-1}} * \frac{\partial}{\partial r_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})}.$$

Similarly, we also get

$$\frac{\partial}{\partial \theta} \Big|_{\tilde{\pi}(e^z \widehat{W})} = \mathbb{L}_{g^{-1}} * \frac{\partial}{\partial \theta_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})}.$$

Then,

$$\begin{aligned} & \omega_{(X,g)} \left( \frac{\partial}{\partial r_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})}, \frac{\partial}{\partial \theta_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})} \right) \\ &= \omega_{(X,g)} \left( \tilde{\pi}_* L_{g*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, \tilde{\pi}_* L_{g*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right) \\ &= \det \begin{pmatrix} \langle L_{(ge^z \widehat{W})^{-1}*} L_{g*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, \widehat{W} \rangle & \langle L_{(ge^z \widehat{W})^{-1}*} L_{g*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}}, \widehat{W} \rangle \\ \langle L_{(ge^z \widehat{W})^{-1}*} L_{g*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, i\widehat{W} \rangle & \langle L_{(ge^z \widehat{W})^{-1}*} L_{g*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}}, i\widehat{W} \rangle \end{pmatrix} \\ &= \det \begin{pmatrix} \langle L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, \widehat{W} \rangle & \langle L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}}, \widehat{W} \rangle \\ \langle L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, i\widehat{W} \rangle & \langle L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}}, i\widehat{W} \rangle \end{pmatrix} \\ &= \omega_0 \left( \frac{\partial}{\partial r} \Big|_{\tilde{\pi}(e^z \widehat{W})}, \frac{\partial}{\partial \theta} \Big|_{\tilde{\pi}(e^z \widehat{W})} \right) \\ &= \omega_0 \left( \mathbb{L}_{g^{-1}} * \frac{\partial}{\partial r_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})}, \mathbb{L}_{g^{-1}} * \frac{\partial}{\partial \theta_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})} \right) \\ &= \mathbb{L}_{g^{-1}} * \omega_0 \left( \frac{\partial}{\partial r_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})}, \frac{\partial}{\partial \theta_1} \Big|_{\tilde{\pi}(ge^z \widehat{W})} \right). \end{aligned}$$

To prove part (iii), consider the induced area form  $\omega$  on  $S$  from the metric on  $D_{n,m}$ . Since the restriction  $\mathbb{L}_{g^{-1}} : S \rightarrow S_0$  of  $\mathbb{L}_{g^{-1}} : D_{n,m} \rightarrow D_{n,m}$  is an isometry, we get

$$\omega = \mathbb{L}_{g^{-1}}^* \omega_1$$

for the induced area form  $\omega_1$  on  $S_0$  from the metric on  $D_{n,m}$ . And, under the notation  $x^h$  of the horizontal part of a tangent vector  $x \in T(U(n,m))$  with respect to the Riemannian submersion  $\tilde{\pi} : U(n,m) \rightarrow D_{n,m}$ , Equations (4-2) and (4-3) show that

$$L_{e^{-z} \widehat{W}*} \left( \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right)^h = \left( L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right)^h = L_{e^{-z} \widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}$$

and that

$$\begin{aligned}
L_{e^{-z}\widehat{W}*} \left( \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right)^h &= \left( L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right)^h \\
&= \begin{pmatrix} O_n & \frac{-i}{2} e^{-i\theta} \sum_{k=0}^{\infty} \frac{(2r)^{2k+1}}{(2k+1)!} (W^* W)^k W^* \\ \frac{i}{2} e^{i\theta} \sum_{k=0}^{\infty} \frac{(2r)^{2k+1}}{(2k+1)!} W (W^* W)^k & O_m \end{pmatrix} \\
&= \begin{pmatrix} O_n & \frac{-i}{2} e^{-i\theta} \sum_{k=0}^{\infty} \frac{(2r)^{2k+1}}{(2k+1)!} A (\Sigma^* \Sigma)^k \Sigma^* B^* \\ \frac{i}{2} e^{i\theta} \sum_{k=0}^{\infty} \frac{(2r)^{2k+1}}{(2k+1)!} B \Sigma (\Sigma^* \Sigma)^k A^* & O_m \end{pmatrix} \\
&= \Omega \begin{pmatrix} O_n & \frac{-i}{2} e^{-i\theta} \Upsilon(z)^* \\ \frac{i}{2} e^{i\theta} \Upsilon(z) & O_m \end{pmatrix} \Omega^*,
\end{aligned}$$

where

$$\Upsilon_a(z) = \text{diag}[\sinh(2\sigma_1 r) \cdots, \sinh(2\sigma_a r)],$$

and

$$\Upsilon(z) = \begin{cases} \Upsilon_a & \text{in case of } n = m, \\ (\Upsilon_a \ O)^T \in M_{m \times n} & \text{in case of } n < m, \\ (\Upsilon_a \ O) \in M_{m \times n} & \text{in case of } n > m. \end{cases}$$

Then,

$$\begin{aligned}
\left| L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right| &= \sqrt{\frac{1}{2} \text{Re} \left( \text{Tr} \left( \left( L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right)^* L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right) \right)} \\
&= \sqrt{\sigma_1^2 + \cdots + \sigma_n^2} \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
&2 \langle L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}}, L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \rangle \\
&= \text{Re} \left( \text{Tr} \left( \left( L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial r} \Big|_{e^z \widehat{W}} \right)^* L_{e^{-z}\widehat{W}*} \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right) \right) \\
&= \text{Re} \left( \text{Tr} \left( \Omega \begin{pmatrix} O_n & e^{-i\theta} \Sigma^* \\ e^{i\theta} \Sigma & O_m \end{pmatrix} \begin{pmatrix} O_n & \frac{-i}{2} e^{-i\theta} \Upsilon(z)^* \\ \frac{i}{2} e^{i\theta} \Upsilon(z) & O_m \end{pmatrix} \Omega^* \right) \right) \\
&= \text{Re} \left( \text{Tr} \left( \begin{pmatrix} \frac{i}{2} \Sigma^* \Upsilon(z) & O \\ O & \frac{-i}{2} \Sigma \Upsilon(z)^* \end{pmatrix} \right) \right) \\
&= 0
\end{aligned}$$



and

$$\begin{aligned}
& \left| \left( L_{e^{-z}\widehat{W}} \ast \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right)^h \right| \\
&= \sqrt{\frac{1}{2} \operatorname{Re} \left( \operatorname{Tr} \left( \left( \left( L_{e^{-z}\widehat{W}} \ast \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right)^h \right)^* \left( L_{e^{-z}\widehat{W}} \ast \frac{\partial}{\partial \theta} \Big|_{e^z \widehat{W}} \right)^h \right) \right)} \\
&= \sqrt{\frac{1}{8} \operatorname{Re} \left( \operatorname{Tr} \left( \begin{pmatrix} \Upsilon(z)^* \Upsilon(z) & O \\ O & \Upsilon(z) \Upsilon(z)^* \end{pmatrix} \right) \right)} \\
&= \frac{1}{2} \sqrt{\sum_{j=1}^n \sinh^2(2\sigma_j r)},
\end{aligned}$$

which show the former one of the part (iii).

To show the latter one, assume  $\sigma_1 = \cdots = \sigma_q > 0 = \sigma_{q+1} = \cdots = \sigma_n$ . Then,  $\sigma_1 = \frac{1}{\sqrt{q}}$  from  $\sigma_1^2 + \cdots + \sigma_n^2 = 1$  and

$$\begin{aligned}
\omega_1 &= \frac{\sqrt{q}}{2} \sinh\left(\frac{2}{\sqrt{q}}r\right) dr \wedge d\theta \\
&= \sqrt{q} \sinh\left(\frac{1}{\sqrt{q}}r\right) \cosh\left(\frac{1}{\sqrt{q}}r\right) dr \wedge d\theta \\
&= d\left(\frac{q}{2} \sinh^2\left(\frac{1}{\sqrt{q}}r\right)\right) d\theta \\
&= d\left(\sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r) d\theta\right) \\
&= \omega_0.
\end{aligned}$$

And

$$W^*W = A\Sigma^*\Sigma A^* = A \operatorname{diag}[\underbrace{\sigma_1^2, \dots, \sigma_1^2}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}}] A^*$$

shows that

$$\begin{aligned}
\det(xI_n - W^*W) &= \det\left(A(xI_n - \operatorname{diag}[\underbrace{\sigma_1^2, \dots, \sigma_1^2}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-q) \text{ times}}])A^*\right) \\
&= (x - \sigma_1^2)^q x^{(n-q)}.
\end{aligned}$$

## 5. PROOF OF THEOREM 1.5

We follow the notation in the proof of Proposition 1.4.

Let  $K = U(m)$ .

Note that the equations

$$\tilde{\pi} = \pi \circ \hat{\pi}, \quad \hat{\pi} \circ L_g = \widehat{L}_g \circ \hat{\pi} \quad \text{and} \quad \tilde{\pi} \circ L_g = \mathbb{L}_g \circ \tilde{\pi}$$

give

$$\pi(\widehat{L}_g(\hat{\pi}(\tilde{S}))) = S = \mathbb{L}_g(S_0).$$

And for any  $g \in U(n, m)$  and for any  $h_1 \in U(n)$ , both  $\widehat{L}_g, \widehat{R}_{h_1} : U(n, m)/U(m) \rightarrow U(n, m)/U(m)$  are isometries preserving the fibers of the bundle  $\pi : U(n, m)/U(m) \rightarrow D_{n, m}$  such that

$$\pi \circ \widehat{L}_g = \mathbb{L}_g \circ \pi \quad \text{and} \quad \pi \circ \widehat{R}_{h_1} = \pi.$$

Thus the curve  $c$  on  $S$  can be written by  $\mathbb{L}_g \circ c_1$  for some curve  $c_1$  on  $S_0$ , and then for a horizontal lift  $\widehat{c}_1$  of  $c_1$  on  $U(n, m)/U(m)$ , the horizontal lift  $\widehat{c}$  of  $c = \mathbb{L}_g \circ c_1$  will be given by  $\widehat{R}_{h_1} \circ \widehat{L}_g \circ \widehat{c}_1$  for some  $h_1 \in U(n)$ . Note that if

$$\widehat{c}(1) = \widehat{c}(0) \cdot h_0, \quad h_0 \in U(n)$$

then

$$(\widehat{R}_{h_1} \circ \widehat{L}_g \circ \widehat{c}_1)(1) = (\widehat{R}_{h_1} \circ \widehat{L}_g \circ \widehat{c}_1)(0) \cdot (h_1^{-1} h_0 h_1).$$

And, if  $h_0 = e^\Phi$  for some  $\Phi \in \mathfrak{u}(n)$ , then

$$h_1^{-1} h_0 h_1 = e^{\text{Ad}_{h_1^{-1}} \Phi}$$

and

$$\text{Tr}(\text{Ad}_{h_1^{-1}} \Phi) = \text{Tr}(\Phi).$$

So, Proposition 1.4(ii) enables us to suppose that  $g$  is the identity of  $U(n, m)$  without loss of generality.

Before constructing the horizontal lift  $\widehat{c}(t)$  of  $c(t) \in S_0, t \in [0, 1]$ , in the bundle  $U(n, m)/U(m) \xrightarrow{\pi} D_{n, m}$ , let  $z(t) = r(t)e^{i\theta(t)}$  and  $\widetilde{\gamma}(t) = e^{\widetilde{z(t)}\widehat{W}}$ , the lift of  $c$  in the bundle  $U(n, m) \rightarrow D_{n, m}$ , lying in  $\widetilde{S} \subset U(n, m)$ . From Lemma 2.1, consider the Lie subgroup  $\widehat{G} (\subset U(n, m))$  of the Lie algebra  $\widehat{\mathfrak{g}}$  generated by  $\{\widehat{W}, i\widehat{W}\}$ , which is the same as the one by  $\{\widehat{X}, i\widehat{X}\}$ . Follow the notation of Lemma 2.1 and let  $\widehat{H}$  be the Lie group of the Lie algebra  $\widehat{\mathfrak{h}}$  generated by  $\{\widetilde{V}_k | k = 1, 2, \dots\}$ . Then  $\widehat{H}$  is a subset of the intersection of  $\widehat{G}$  and  $U(n) \times U(m)$ , which means that for each element of  $\widehat{H}$ , the map, given by its right multiplication in  $\widehat{G}$  with the left invariant metric induced from the metric of  $U(n, m)$  determined by (1-1), is an isometry of  $\widehat{G}$ . Consider the inclusion map  $\iota : \widehat{G} \hookrightarrow U(n, m)$ , its Lie algebra homomorphism  $d\iota : \widehat{\mathfrak{g}} \rightarrow \mathfrak{u}(n, m)$  and the Riemannian submersion  $\widehat{p}r : \widehat{G} \rightarrow \widehat{G}/\widehat{H}$ . Since

$$(5-1) \quad d\iota(\widetilde{V}_k) = \widetilde{V}_k, \quad d\iota(\widetilde{X}_k) = \widetilde{X}_k \quad \text{and} \quad d\iota(i\widetilde{X}_k) = i\widetilde{X}_k$$

for  $k = 1, 2, \dots$ , Lemma 2.1, the equation (4-2) on  $L_{e^{-z\widehat{W}}} \frac{\partial}{\partial r} \big|_{e^{z\widehat{W}}}$  and the equation (4-3) on  $L_{e^{-z\widehat{W}}} \frac{\partial}{\partial \theta} \big|_{e^{z\widehat{W}}}$  in the proof of Proposition 1.4 show that  $\widetilde{S}$  is a subset of  $\widehat{G}$ . The tangent vector of any horizontal curve  $\widetilde{c}$  in  $\widehat{G} \rightarrow \widehat{G}/\widehat{H}$  is spanned by  $\widetilde{X}_k$  and by  $i\widetilde{X}_k$ ,  $k = 1, 2, \dots$ . The tangent vector of its induced curve  $\iota \circ \widetilde{c}$  in  $U(n, m)$  is too, from the equation (5-1), which means that, by abusing of notation,  $\iota \circ \widetilde{c} = \widetilde{c}$  is a horizontal curve in the bundle  $U(n, m) \rightarrow D_{n, m}$  since both  $\widetilde{X}_k$  and  $i\widetilde{X}_k$  are still horizontal in this bundle.

In fact,  $\widetilde{V}_k, \widetilde{X}_k$  and  $i\widetilde{X}_k$  in  $\mathfrak{g}$  are different from  $\widetilde{V}_k, \widetilde{X}_k$  and  $i\widetilde{X}_k$  in  $\hat{\mathfrak{g}}$  as tangent vector fields, but they are  $\iota$ -related, *i.e.*,

$$\iota_* \widetilde{V}_k = \widetilde{V}_k \circ \iota, \quad \iota_* \widetilde{X}_k = \widetilde{X}_k \circ \iota \quad \text{and} \quad \iota_* i\widetilde{X}_k = i\widetilde{X}_k \circ \iota,$$

and so  $\widehat{\iota \circ \tilde{c}} = \iota_* \dot{\tilde{c}}$  is generated by  $\widetilde{X}_k, i\widetilde{X}_k$  in  $\mathfrak{g}$ . And the horizontal curve  $\tilde{c}$  in  $\widehat{G}$  with  $\widehat{p}r \circ \tilde{c} = \widehat{p}r \circ \tilde{\gamma}$  can be given by  $\tilde{c}(t) = \tilde{\gamma}(t)e^{\widehat{\Psi}(t)}$  for some curve  $\widehat{\Psi} : [0, 1] \rightarrow \hat{\mathfrak{h}}$  and will be regarded as a horizontal lift of  $c$  in the bundle  $U(n, m) \rightarrow D_{n,m}$ . For

$$\widehat{\Psi}(t) = \begin{pmatrix} \eta_1(t) & O \\ O & \eta_2(t) \end{pmatrix} \in \hat{\mathfrak{h}} \subset \mathfrak{u}(n) + \mathfrak{u}(m),$$

let

$$\Psi(t) = \begin{pmatrix} \eta_1(t) & O \\ O & O_m \end{pmatrix} \in \mathfrak{u}(n) + \{O_m\}.$$

Then, for the curve  $\bar{c}(t) = \tilde{\gamma}(t)e^{\Psi(t)}$  in  $U(n, m)$ , the equation

$$\tilde{c}(t)K = \tilde{\gamma}(t)e^{\widehat{\Psi}(t)}K = \tilde{\gamma}(t)e^{\Psi(t)}K = \bar{c}(t)K$$

holds. Therefore, to construct the horizontal lift  $\hat{c}(t) = \tilde{c}(t)K = \bar{c}(t)K$  of  $c(t)$  in the bundle  $U(n, m)/U(m) \xrightarrow{\pi} D_{n,m}$ , it suffices to find such a curve  $\Psi(t) \in \mathfrak{u}(n)$ .

Note that

$$(5-2) \quad \tilde{\gamma}(t) = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \Gamma_n(t) & \Lambda(t)^* \\ \Lambda(t) & \Gamma_m(t) \end{pmatrix} \begin{pmatrix} A^* & O \\ O & B^* \end{pmatrix}$$

$$(5-3) \quad = \Omega \begin{pmatrix} \Gamma_n(t) & \Lambda(t)^* \\ \Lambda(t) & \Gamma_m(t) \end{pmatrix} \Omega^*,$$

where

$$\Gamma_j(t) = \text{diag}[\cosh(\sigma_1 r(t)), \dots, \cosh(\sigma_j r(t))], \quad j = n, m$$

and for  $\Lambda_a(t) = \text{diag}[e^{i\theta(t)} \sinh(\sigma_1 r(t)), \dots, e^{i\theta(t)} \sinh(\sigma_a r(t))]$ ,

$$\Lambda(t) = \begin{cases} \Lambda_a(t) & \text{in case of } a = n = m, \\ (\Lambda_a(t) \ O)^T \in M_{m \times n} & \text{in case of } a = n < m, \\ (\Lambda_a(t) \ O) \in M_{m \times n} & \text{in case of } a = m < n. \end{cases}$$

Lemma 2.1,  $W^*W = \frac{1}{|\widehat{X}|^2} X^*X$  and  $WW^* = \frac{1}{|\widehat{X}|^2} XX^*$  enable us to consider a curve

$$\widehat{\Psi}(t) = \begin{pmatrix} i \sum_{k=1}^q \phi_k(t)(W^*W)^k & O \\ O & -i \sum_{k=1}^q \phi_k(t)(WW^*)^k \end{pmatrix} \in \hat{\mathfrak{h}}$$

such that  $q = \text{rk}W$  and that

$$\tilde{c}(t) = \tilde{\gamma}(t)e^{\widehat{\Psi}(t)}.$$

Let

$$\Psi(t) = \begin{pmatrix} i \sum_{k=1}^q \phi_k(t)(W^*W)^k & O \\ O & O_m \end{pmatrix} \in \mathfrak{u}(n).$$

Then, by abusing of notations, we can say that  $\Psi(t) \in \text{Span}_{\mathbb{R}}\{i(X^*X)^k\}_{k=1}^q$ .  
And

$$\Psi(t) = \Omega \text{diag} \left[ i \sum_{k=1}^q \sigma_1^{2k} \phi_k(t), \dots, i \sum_{k=1}^q \sigma_n^{2k} \phi_k(t), \underbrace{0, \dots, 0}_{m\text{-times}} \right] \Omega^*.$$

So, we get

$$(5-4) \quad e^{\Psi(t)} = \Omega \text{diag} \left[ e^{i \sum_{k=1}^q \sigma_1^{2k} \phi_k(t)}, \dots, e^{i \sum_{k=1}^q \sigma_n^{2k} \phi_k(t)}, \underbrace{1, \dots, 1}_{m\text{-times}} \right] \Omega^*$$

and

$$\hat{c}(t) = \tilde{\gamma}(t) e^{\hat{\Psi}(t)} K = \tilde{\gamma}(t) e^{\Psi(t)} K = \bar{c}(t) K$$

for  $\bar{c}(t) = \tilde{\gamma}(t) e^{\Psi(t)}$ . Note

$$\begin{aligned} & L_{\bar{c}(t)^{-1}} \dot{\bar{c}}(t) + \mathfrak{u}(m) \\ &= e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} (\tilde{\gamma}'(t) e^{\Psi(t)} + \tilde{\gamma}(t) e^{\Psi(t)} \Psi'(t)) + \mathfrak{u}(m) \\ &= (e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} \tilde{\gamma}'(t) e^{\Psi(t)} + \Psi'(t)) + \mathfrak{u}(m), \end{aligned}$$

and the direct calculation  $e^{-\Psi(t)} \tilde{\gamma}(t)^{-1} \tilde{\gamma}'(t) e^{\Psi(t)} + \Psi'(t)$  through the singular value decompositions (5-2) and (5-4) of  $\tilde{\gamma}$  and  $e^{\Psi(t)}$  says that its first  $(n \times n)$ -block matrix is

$$A \text{diag} \left[ i \left( \sum_{k=1}^q \sigma_j^{2k} \phi'_k(t) - \theta'(t) \sinh^2(\sigma_j r(t)) \right) \right]_{j=1}^n A^*.$$

Note that  $\sigma_{q+1} = \dots = \sigma_n = 0$  if  $n > q = \text{rk} W$ .

Then  $\hat{c}(t) = \bar{c}(t) K$  is a horizontal curve in  $G/K$  if and only if

$$\sum_{k=1}^q \sigma_j^{2k} \phi'_k(t) = \theta'(t) \sinh^2(\sigma_j r(t)), \quad j = 1, \dots, q.$$

Therefore, Proposition 1.4 says that, for  $\Psi := \Psi(1) - \Psi(0) \in \text{Span}_{\mathbb{R}}\{i(X^*X)^k\}_{k=1}^q \subset \mathfrak{u}(n)$  and for the region  $D(\subset S_0)$  enclosed by the

given orientation-preserving curve  $c : [0, 1] \rightarrow S_0$ ,

$$\begin{aligned}
\mathrm{Tr}(\Psi) &= i \sum_{j=1}^n \sum_{k=1}^q \sigma_j^{2k} (\phi_k(1) - \phi_k(0)) \\
&= i \int_0^1 \sum_{j=1}^n \theta'(t) \sinh^2(\sigma_j r(t)) dt \\
&= 2i \int_{[0,1]} \sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r(t)) \theta'(t) dt \\
&= 2i \int_c \sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r) d\theta \\
&= 2i \int_D d\left(\sum_{j=1}^n \frac{1}{2} \sinh^2(\sigma_j r) d\theta\right) \\
&= 2i \int_D \omega_0 \\
&= 2i \mathrm{Area}(c).
\end{aligned}$$

And,  $\tilde{\gamma}(0) = \tilde{\gamma}(1)$  and the Lie bracket  $[i(X^*X)^k, i(X^*X)^j] = O_n$  in  $\mathfrak{u}(n)$  say that

$$\bar{c}(0)^{-1} \bar{c}(1) = (\tilde{\gamma}(0) e^{\Psi(0)})^{-1} (\tilde{\gamma}(1) e^{\Psi(1)}) = e^{-\Psi(0)} e^{\Psi(1)} = e^{\Psi}.$$

Since the holonomy displacement is given by the right action of  $e^{\Psi} \in U(n)$ , *i.e.*,

$$\hat{c}(1) = \bar{c}(1)K = (\bar{c}(0) e^{\Psi}) K = \bar{c}(0) K \cdot e^{\Psi} = \hat{c}(0) \cdot e^{\Psi},$$

the theorem is proved. □

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